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COMPARATIVE ANALYSIS OF THE TWO-CONSTANT GENERALIZATIONS OF HOOKE'S LAW FOR ISOTROPIC ELASTIC MATERIALS AT FINITE STRAINS

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For ten models of the isothermal behavior of materials, the solutions of boundary-value problems are studied for five types of the experimentally reproducible uniform stress–strain state with unchanged directions of the principal axes. It is found that, for three models, the governing equations are similar to the relations of Hooke's law and valid within the same range of the ratio between the shear and bulk moduli. In these models, the specific strain energy can be represented as a sum of the energies due to changes in volume and shape. The ranges where the other three known models exhibit incorrect behavior are determined.

Introduction. It is well known that large reversible strains occur mainly in elastomers and amorphous polymers in a highly elastic state. Experimental data on mechanical loading of these materials show that they possess viscoelastic and thixotropic properties. For practical purposes, these effects can be ignored, and equations of state of an elastic body can be used as a first (equilibrium) approximation. Elastic models are also the basis for constructing more complex governing equations.

For an isotropic body undergoing small isothermal strains, Hooke's law contains only two Lamé constants μ and λ (μ is the shear modulus) with allowable values $\mu > 0$ and $3\lambda + 2\mu > 0$. Truesdell [1] states that these inequalities are necessary and sufficient for any shear strain to cause the shear stress of the same sign and for a local volume to increase or decrease depending on whether the mean stress is positive or negative. They ensure the positive work done in any infinitesimal strain from a natural stress-free configuration and the existence and uniqueness of the solution. They are also sufficient for the velocities of propagation of waves of all kinds to be real numbers.

Variety of finite-strain measures has led to various generalizations of Hooke's law for subsets of the above-mentioned range of variability of the Lamé parameters (incompressible, slightly compressible, and compressible materials). At present, the complete system of supplementary inequalities [1–7] similar in rigor and sufficiency to the above inequalities for the Lamé parameters is lacking. For example, the models of a polylinear body and Murnaghan's linear material [3] used in theoretical applications satisfy Hadamard's conditions [5, 6] but predict a physically incorrect change in volume [8].

This study aims to determine simplest equations of state that describe satisfactorily the experimental data in a wide range of elastic constants for strains varying from -0.5 to 1 for different types of the stress–strain state. In contrast to [9], attention is focused on the description of the volume strain as one of the main criteria in choosing the equations of state.

Types of the Stress–Strain State. We consider some cases of uniform deformation with invariable directions of the principal axes of strain, which are coaxial with the principal axes of the Cauchy stress tensor T. We choose the Cartesian coordinate system as a reference frame for the undeformed configuration of a body. In this case, the idealized specimen of a material shaped as a cube with edges of unit length and faces perpendicular to the directions of the principal axes of the tensor T deforms into a parallelepiped with edges of length λ_i (hereafter $i = 1$, 2, and 3), which completely determines its state.

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Models of elastic and hyperelastic materials are distinguished [3]. For elastic materials, the stresses in the specimen are directly determined by the principal values of the Cauchy stress tensor t_i. For incompressible hyperelastic materials determined by a potential W, we have [3, 6]

$$t_i = \lambda_i \frac{\partial W}{\partial \lambda_i} + p = 2\lambda_i^2 \frac{\partial W}{\partial I_1} - 2\lambda_i^{-2} \frac{\partial W}{\partial I_2} + p,$$

where p is the undetermined Lagrange multiplier; for compressible hyperelastic materials, we obtain

$$t_i = \lambda_i I_3^{-1/2} \frac{\partial W}{\partial \lambda_i} = 2\lambda_i^2 I_3^{-1/2} \Big(\frac{\partial W}{\partial I_1} + (I_1 - \lambda_i^2) \frac{\partial W}{\partial I_2} \Big) + 2I_3^{1/2} \frac{\partial W}{\partial I_3},$$

where I_i are the principal invariants of the Cauchy–Green strain measure, which are equal to the invariants of Finger's strain measure: $I_1 = \lambda_1^2 + \lambda_2^2 + \lambda_3^2$, $I_2 = \lambda_1^2\lambda_2^2 + \lambda_2^2\lambda_3^2 + \lambda_3^2\lambda_1^2$, and $I_3 = \lambda_1^2\lambda_2^2\lambda_3^2$.

As in [9], we consider five types of the stress–strain state:

1) Hydrostatic stress state (HSS): $\lambda_1 = \lambda_2 = \lambda_3$ and $t_i = -q$ (q is the hydrostatic pressure);

2) Uniaxial deformed state (UDS): $\lambda_1 = \lambda_2 = 1$ and $t_1 = t_2$;

3) Uniaxial stress state (USS): $\lambda_1 = \lambda_2$ and $t_1 = t_2 = 0$;

4) Symmetric biaxial stress state (SBSS): $\lambda_1 = \lambda_2$, $t_1 = t_2$, and $t_3 = 0$;

5) Nonsymmetric biaxial deformed state (NBDS): $\lambda_2 = 1$ and $t_3 = 0$ (in [9], the symmetric biaxial deformed state is analyzed instead).

Generalizations of Hooke's Law. We consider models of incompressible materials ($I_3 \equiv 1$) with the following potentials:

— Treloar's potential ("neo-Hookean body") [10]

$$W_1 = (1/2)\mu(I_1 - 3); \tag{1}$$

— Bartenev–Khazanovich potential [11]

$$W_2 = 2\mu(\lambda_1 + \lambda_2 + \lambda_3 - 3) \tag{2}$$

and the model of Hencky's incompressible material [3] with zero angle of deviatoric similarity

$$T_3 = 2\mu N + pE, \tag{3}$$

where N is the logarithmic-strain measure tensor determined in the basis of the principal axes of strain and E is the unit tensor.

The models of the following compressible materials are considered:

— Hencky's material [3] with a zero phase of deviatoric similarity, which coincides with (3) for $I_3 = 1$:

$$T_4 = 2\mu N + \lambda \ln(\sqrt{I_3})E; \tag{4}$$

— Murnaghan's linear material determined by two first terms of the expansion of W in powers of the Cauchy–Green strain tensor invariants [3]:

$$W_5 = (1/8)(\lambda + \mu)(I_1 - 3)^2 - (1/2)\mu(I_2 - 2I_1 + 3); \tag{5}$$

— polylinear material (harmonic material and John's material) [3]

$$W_6 = (1/2)\lambda(\delta_1 + \delta_2 + \delta_3)^2 + \mu(\delta_1^2 + \delta_2^2 + \delta_3^2) \tag{6}$$

($\delta_i = \lambda_i - 1$ are the principal relative extensions);

— material described by the simplified Signorini's law [3]

$$W_7 = \sqrt{I_3}\Big[(1/2)(\lambda + \mu)(j_1^\times)^2 + \mu(1 - j_1^\times)\Big] - \mu \tag{7}$$

[$j_1^\times = A_1 + A_2 + A_3 = (3 - I_2/I_3)/2$ is the first invariant of the Almansi strain tensor, where $A_i = (1 - \lambda_i^{-2})/2$ are its principal values].

For slightly compressible materials, we consider the variants of the general two-constant potentials:

— Variant of the Peng–Landel potential [12], which coincides with (1) for $I_3 = 1$:

$$W_8 = (1/2)\mu(I_1 I_3^{-1/3} - 3) + (1/2)B\Theta^2 \tag{8}$$

($B = \lambda + 2\mu/3$ is the parameter which becomes the bulk modulus for small strains and $\Theta = \sqrt{I_3} - 1$ is the relative change in volume);

TABLE 1

Model	$t_1 = t_2 = t_3$
Hooke's law	$3B\delta_1$
T_4	$3B \ln \lambda_1$
W_5	$3B(\lambda_1^2 - 1)\lambda_1^{-1}/2$
W_6	$3B(\lambda_1 - 1)\lambda_1^{-2}$
W_7	$3BA_1[1 - (3 + \mu/B)A_1/6]$
W_8	$B\Theta$
W_9	$B\Theta$
W_{10}	$B(I_3 - 1)\lambda_1^3/2 + \mu(3\lambda_1^{-1} - 5\lambda_1^3 + 2\lambda_1^9)/3$

— Variant of the Chernykh–Shubina potential [7], which coincides with (2) for $I_3 = 1$:

$$W_9 = 2\mu[(\lambda_1 + \lambda_2 + \lambda_3)I_3^{-1/6} - 3] + (1/2)B\Theta^2; \tag{9}$$

— Variant of Rogovoi's potential [13], which corresponds to the incompressible material (1) for $I_3 = 1$:

$$W_{10} = (1/2)\mu(I_1 - I_3 - 2) + (1/8)(\lambda + 2\mu)(I_3 - 1)^2. \tag{10}$$

The materials described by (1)–(10) belong to the class of simple materials [1]. The constants that enter these potentials coincide with the constants in Hooke' law in the limiting case, and they can be determined experimentally for uniform stress–strained states.

According to [7], models (1)–(7) can be classified by the generalized strain tensors of the real order n of strain measure: models (1)–(3) refer to incompressible "standard" materials of the second, first, and zeroth order, respectively, and models (4)–(7) refer to compressible "standard" materials of the zeroth, second, first, and minus second order, respectively. The models of slightly compressible materials (8) and (9) are particular, simplest cases of the second and first order of the governing equation [14], respectively, in which the distortion tensor is used.

In some cases, the use of strain measures with fractional indices n in constructing models of real materials [15] simplify the governing equations, but the resulting generalizations of Hooke's law involve the index n as an additional parameter of the equation of state. In (1)–(10), only some of its integer values are used.

Discussion of Results. In solving the boundary-value problems for USS, SBSS, and NBDS, the use of the natural boundary conditions allows one to express the experimentally measured stresses by formulas that contain only the parameter μ and are stable with respect to the errors in determining the strain components. A numerical analysis of the corresponding boundary-value problems is performed for various values of the ratio μ/B. The maximum value $\mu/B = 1$ considered characterizes conventionally called porous materials [in the linear theory, Poisson's ratio is $\nu = (3B - 2\mu)/(6B + 2\mu) = 0.125$] and the minimum value $\mu/B = 0.001$ ($\nu = 0.4995$) characterizes compact (monolithic) materials.

Rogovoi [13] showed that model (10) is applicable only to slightly compressible materials. The validity of this statement is supported by a numerical analysis of the changes in volume for all types of the stress–strain state considered for $\mu/B = 1$. Therefore, model (10) is analyzed only in the region of weak compressibility ($\mu/B \ll 1$).

Hydrostatic Stress State. Table 1 lists the dependences of the principal values of the stress tensor ($t_1 = t_2 = t_3$) for Hooke's law and models (4)–(10) on the constants and principal relative extension λ_1. One can see that the dependences for models (8) and (9) coincide. The graphs of the dependences for models (4) and (5) and also models (6) and (7) almost coincide for real pressures and $\mu/B = 1.0$–0.001. In contrast to the other models, model (10) gives a "soft" (with a negative second derivative) dependence $t_1(\Theta)$ in the region of compression.

For models (7) and (10), the dependence of $t_1(\Theta)$ on the parameter μ is pronounced only for finite strains; therefore, for models (4)–(10), the parameter B can be determined directly from the initial segment of the curve $t_1(\Theta)$ without allowance for μ.

Uniaxial Deformed State. Table 2 shows the dependences $t_1(\lambda_3) = t_2(\lambda_3)$ and $t_3(\lambda_3)$. For models (5) and (10), the dependences $t_3(\lambda_3)$ coincide, whereas the dependences $t_1(\lambda_3)$ are different. Comparing the dependences $t_3/B \sim \Theta$ for UDS with the dependence $t_1/B \sim \Theta$ for HSS in the same range of stresses, we infer that nonlinearity of the curves $t_3/B \sim \Theta$ is more pronounced.

TABLE 2

Model	$t_1 = t_2$	t_3
Hooke's law	$\lambda\delta_3$	$(\lambda + 2\mu)\delta_3$
T_4	$\lambda \ln \lambda_3$	$(\lambda + 2\mu) \ln \lambda_3$
W_5	$\lambda(\lambda_3^2 - 1)\lambda_3^{-1}/2$	$(\lambda + 2\mu)(\lambda_3^2 - 1)\lambda_3/2$
W_6	$\lambda(\lambda_3 - 1)\lambda_3^{-1}$	$(\lambda + 2\mu)(\lambda_3 - 1)$
W_7	$\lambda A_3 + (\lambda + \mu)A_3^2/2$	$(\lambda + 2\mu)A_3 - 3(\lambda + \mu)A_3^2/2$
W_8	$B\Theta + \mu I_3^{-5/6}(\lambda_3^2 - 1)/3$	$B\Theta + 2\mu I_3^{-5/6}(\lambda_3^2 - 1)/3$
W_9	$(B - 2\mu\lambda_3^{-4/3}/3)\Theta$	$(B + 4\mu\lambda_3^{-4/3}/3)\Theta$
W_{10}	$(\lambda + 2\mu)(I_3 - 1)\lambda_3/2 + \mu(\lambda_3^{-1} - \lambda_3)$	$(\lambda + 2\mu)(\lambda_3^2 - 1)\lambda_3/2$

TABLE 3

Model	Relation $\lambda_1 \sim \lambda_3$	t_3
Hooke's law	$\delta_1 = -\nu\delta_3$	$E\delta_3$
W_1	$\lambda_1 = \lambda_3^{-1/2}$	$\mu(\lambda_3^2 - \lambda_1^2)$
W_2	$\lambda_1 = \lambda_3^{-1/2}$	$2\mu(\lambda_3 - \lambda_1)$
T_3	$\lambda_1 = \lambda_3^{-1/2}$	$3\mu \ln \lambda_3 = E \ln \lambda_3$
T_4	$\ln \lambda_1 = -\nu \ln \lambda_3$	$E \ln \lambda_3$
W_5	$\lambda_1^2 - 1 = -\nu(\lambda_3^2 - 1)$	$\mu\lambda_1^{-2}\lambda_3(\lambda_3^2 - \lambda_1^2)$
W_6	$\lambda_1 - 1 = -\nu(\lambda_3 - 1)$	$2\mu\lambda_1^{-2}(\lambda_3 - \lambda_1)$
W_7	$\lambda_1 = (1 + 4\nu A_3 + A_3^2)^{-1/4}$	$2[\mu - (\lambda + \mu)j_1^{\times}](A_3 - A_1)$
W_8	$B\Theta + \mu I_3^{-5/6}(\lambda_1^2 - \lambda_3^2)/3 = 0$	$\mu I_3^{-5/6}(\lambda_3^2 - \lambda_1^2)$
W_9	$B\Theta + 2\mu I_3^{-2/3}(\lambda_1 - \lambda_3)/3 = 0$	$2\mu I_3^{-2/3}(\lambda_3 - \lambda_1)$
W_{10}	$(\lambda + 2\mu)(I_3 - 1) + 2\mu(\lambda_1^{-2}\lambda_3^{-2} - 1) = 0$	$\mu I_3^{-1/2}(\lambda_3^2 - \lambda_1^2)$

For UDS, the rigidity in compression of models (4)–(10) depends on the parameters μ and B in the entire strain range, and therefore, the parameter B can be determined from the compression curve if μ is known. For real strains, the stress state tends to the hydrostatic state as the ratio μ/B decreases. In this case, the difference in the principal stresses is of the order of $2\mu/B$. Therefore, the parameter B can be determined by testing slightly compressible materials in UDS.

Uniaxial Stress State. Table 3 gives the relations between λ_1 and λ_3 obtained from the incompressibility condition or the natural boundary condition $t_1 = 0$ and also the expressions for t_3. Figure 1 shows the volume-change curve $3B\Theta/E \sim \delta_3$ [$E = 9\mu B/(\mu + 3B)$ is the parameter that corresponds to Young's modulus in the linear theory of elasticity] for $\mu/B = 1$ (Fig. 1a) and $\mu/B = 0.001$ (Fig. 1b and c). The numbers at the curves in Figs. 1–3 correspond to the numbers of the model. The character of the curves in Fig. 1c shows that, for $\mu/B = 0.001$, models (5)–(7) describe the volume changes incorrectly.

Analysis of the Dependences $\Theta(\delta_3)$ *for Models* (5)–(7). 1. Murnaghan's Linear Material (5). The volume-change curve $\Theta(\delta_3)$ has a maximum in the range of extension (below, we give expressions for the axial strain and the corresponding bracketed values of δ_3 calculated for $\mu/B = 0.001$) for $\lambda_3 = [1 - 2\mu/(3B)]^{-1/2}$ $(\delta_3 = 3.335 \cdot 10^{-4})$, then, it changes the sign for $\lambda_3 = [2.25 + 6\mu/(3B - 2\mu)]^{1/2} - 0.5$ $(\delta_3 = 6.67 \cdot 10^{-4})$ and reaches the limiting, physically admissible value $\Theta = -1$ for $\lambda_3 = (1 + 1/\nu)^{1/2}$ $(\delta_3 = 0.7326)$, which corresponds to zero volume of the specimen and infinitely high "true" stresses t_3. For $\delta_3 > (1 + 1/\nu)^{1/2}$, the cross-sectional area of the specimen and its volume become negative.

2. Polylinear Material (6). The volume-change curve $\Theta(\delta_3)$ has a maximum in the region of extension for $\delta_3 = (1 - 2\nu)/(3\nu)$ $(\delta_3 = 6.673 \cdot 10^{-4})$, changes the sign for $\delta_3 = 1.5 + (1 - 2\nu)/\nu - [2.25 + (1 - 2\nu)/\nu]^{1/2}$ $(\delta_3 = 13.348 \cdot 10^{-4})$, and reaches the minimum $\Theta = -1$ (zero volume) for $\delta_3 = 1/\nu$ $(\delta_3 = 2.002)$. At this point, the curve of "true" stresses has a discontinuity of the second kind, but the cross-sectional area and volume of the specimen are positive for $\delta_3 > 1/\nu$.

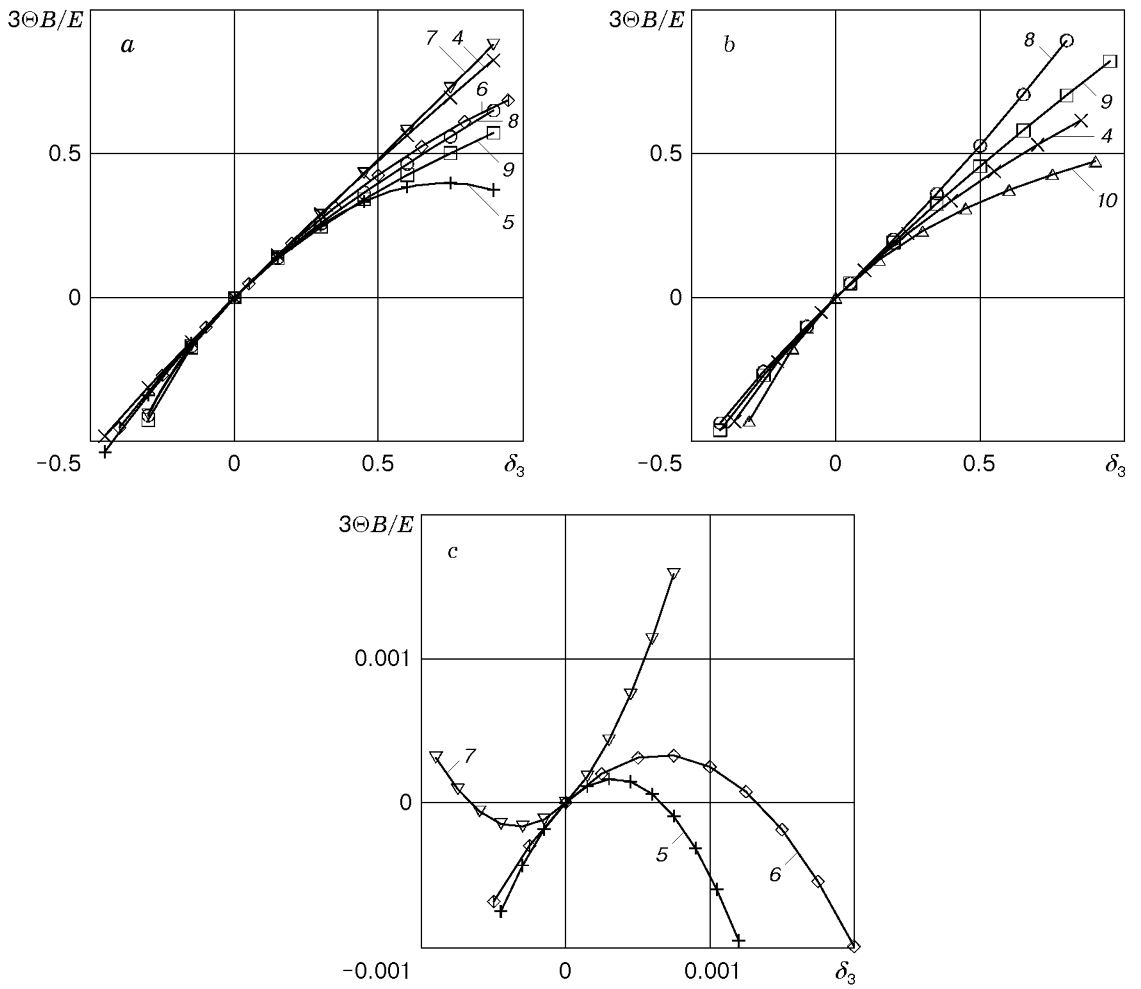

Fig. 1

3. Simplified Signorini's Material (7). The volume-change curve is nonmonotonic in two ranges of μ/B: $\nu > (177^{1/2} - 1)/32 \simeq 0.3845$ $(\mu/B < 0.2503)$ and $\nu < -(177^{1/2} + 1)/32 \simeq -0.4470$ $(\mu/B > 5.1375)$. The first range is of practical interest. In this region, the curve $\Theta(\delta_3)$ reaches a minimum for $A_3 = [1 - 8\nu + (64\nu^2 + 8\nu - 11)^{1/2}]/6$ $(\delta_3 = -3.333 \cdot 10^{-4})$ as the compression increases, and then it increases and has a maximum for $A_3 = [1 - 8\nu - (64\nu^2 + 8\nu - 11)^{1/2}]/6$ $(\delta_3 = -0.4223)$.

Figure 2a shows the dimensionless dependences $t_3^*/E \sim \delta_3$ for $\mu/B = 0.001$, where $t_3^* = t_3\lambda_1^2$ is the component of the "conventional" stress $(t_i^* = t_i I_3^{1/2}/\lambda_i)$ normalized to the cross-sectional area in the undeformed configuration, which is independent of the description of volume changes.

For each material, two dependences listed in Table 3 allow one to determine formally both independent parameters of any model considered. For incompressible materials, they make it possible to estimate the accuracy to which the condition $\lambda_1^2\lambda_3 = 1$ is satisfied and to determine μ in approximating the curve $t_3(\lambda_3)$.

For slightly compressible materials under USS, determination of the parameters B, λ, and ν requires high accuracy in measuring the transversal and longitudinal extensions (to determine small volume changes exactly). Therefore, $\lambda_1(\lambda_3)$ is calculated approximately with the use of the incompressibility condition. Given the dependence of $\lambda_1(\lambda_3)$, the parameter μ (or the parameter $E = 3\mu$ for the hypothesis of incompressibility accepted) in (4) and (8)–(10) is calculated in the same manner as for incompressible materials. This approach leads to relative errors of the order of $2\mu/B$ in approximating $t_3(\lambda_3)$. For SBSS and NBDS, the parameters of the model can be determined in a similar manner. However, these stress states are rarely reproduced in experiments.

Symmetric Biaxial Stress State. The relations between λ_3 and λ_1 and the formulas for stresses $t_1 = t_2$ are listed in Table 4. The results of numerical analysis for $\mu/B = 0.001$ are similar to those obtained in analyzing

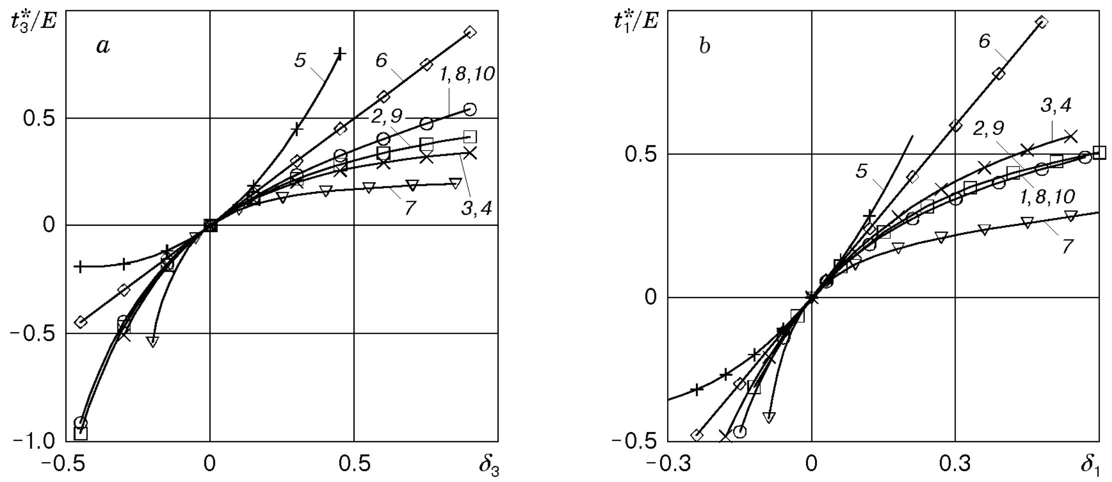

Fig. 2

TABLE 4

Model	Relation $\lambda_3 \sim \lambda_1$	$t_1 = t_2$
Hooke's law	$\delta_3 = -2\lambda\delta_1/(\lambda + 2\mu)$	$6\mu B\delta_1/(\lambda + 2\mu)$
W_1	$\lambda_3 = \lambda_1^{-1/2}$	$\mu(\lambda_1^2 - \lambda_3^2)$
W_2	$\lambda_3 = \lambda_1^{-1/2}$	$2\mu(\lambda_1 - \lambda_3)$
T_3	$\lambda_3 = \lambda_1^{-1/2}$	$2\mu \ln(\lambda_1/\lambda_3)$
T_4	$\ln\lambda_3 = \ln\lambda_1(4\mu - 6B)/(4\mu + 3B)$	$2\mu \ln(\lambda_1/\lambda_3)$
W_5	$\lambda_3^2 - 1 = -2\lambda(\lambda_1^2 - 1)/(\lambda + 2\mu)$	$\mu\lambda_3^{-1}(\lambda_1^2 - \lambda_3^2)$
W_6	$\lambda_3 - 1 = -2\lambda(\lambda_1 - 1)/(\lambda + 2\mu)$	$2\mu\lambda_1^{-1}\lambda_3^{-1}(\lambda_1 - \lambda_3)$
W_7	$[2\lambda/(\lambda+\mu)]j_1^\times + (j_1^\times)^2 + 4[\mu/(\lambda+\mu) - j_1^\times]A_3 = 0$	$2[\mu - (\lambda+\mu)j_1^\times](A_1 - A_3)$
W_8	$B\Theta + 2\mu I_3^{-5/6}(\lambda_1^2 - \lambda_3^2)/3 = 0$	$\mu I_3^{-5/6}(\lambda_1^2 - \lambda_3^2)$
W_9	$B\Theta + 4\mu I_3^{-2/3}(\lambda_1 - \lambda_3)/3 = 0$	$2\mu I_3^{-2/3}(\lambda_1 - \lambda_3)$
W_{10}	$(\lambda + 2\mu)(I_3 - 1) + 2\mu(\lambda_1^{-4} - 1) = 0$	$\mu I_3^{-1/2}(\lambda_1^2 - \lambda_3^2)$

USS: for model (5), no real roots δ_3 were found in the region $\delta_1 > 0.24$; the volume-change curves $3B\Theta/E \sim \delta_1$ for models (5)–(7) are similar to the corresponding curves $3B\Theta/E \sim \delta_3$ for USS. Figure 2b shows the dependences $t_1^*/E \sim \delta_1$ for $\mu/B = 0.001$, from which one can see that the nonlinear behavior for USS differs much from that for SBSS (in the linear model, these dependences differ in a scaling factor only).

Nonsymmetric Biaxial Deformed State. For incompressible materials, NBDS is identified with pure shear [10]. Table 5 shows the relations between λ_3 and λ_1 and the formulas for t_1 and t_2. As in the case of USS, model (5) has no real roots δ_3 in the region $\delta_1 > 0.42$ for $\mu/B = 0.001$. For models (5)–(7), the curves $\Theta(\delta_1)$ are similar.

For NBDS, one can obtain much more information for choosing models by measuring both nonzero stress components in experiments, since the ratio t_1^*/t_2^* depends strongly not only on μ/B and δ_1 but also on the strain measure used. Figure 3 shows the dependences $t_1^*/t_2^* \sim \delta_1$ calculated for different models for $\mu/B = 1$ (Fig. 3a) and $\mu/B = 0.001$ (Fig. 3b). In Fig. 3b, we do not show the curves for models (5)–(7) for the reasons mentioned above.

For the above-considered incompressible materials (and other "standard" incompressible materials [7]), the order n can be determined by the formula

$$n = \frac{d(t_1/t_2)}{d\delta_1}\Big|_{\delta_1=0}.$$

For models (4) and (8)–(10), this formula gives an error proportional to μ/B if $\mu/B \ll 1$. Chernykh and Litvinenkova [7] propose to determine n as a parameter of approximation of experimental data by the "straightening coordinate" method.

TABLE 5

Model	Relation $\lambda_3 \sim \lambda_1$	t_1/t_2
Hooke's law	$\delta_3 = -\lambda\delta_1/(\lambda+2\mu)$	$\dfrac{4\mu\delta_1(\lambda+\mu)/(\lambda+2\mu)}{2\mu\delta_1\lambda/(\lambda+2\mu)}$
W_1	$\lambda_3 = \lambda_1^{-1/2}$	$\mu(\lambda_1^2-\lambda_3^2)/[\mu(1-\lambda_3^2)]$
W_2	$\lambda_3 = \lambda_1^{-1/2}$	$2\mu(\lambda_1-\lambda_3)/[2\mu(1-\lambda_3)]$
T_3	$\lambda_3 = \lambda_1^{-1/2}$	$4\mu\ln\lambda_1/(2\mu\ln\lambda_1)$
T_4	$\ln\lambda_3 = \ln\lambda_1(2\mu-3B)/(4\mu+3B)$	$2\mu\ln(\lambda_1/\lambda_3)/[2\mu\ln(1/\lambda_3)]$
W_5	$\lambda_3^2-1 = -\lambda(\lambda_1^2-1)/(\lambda+2\mu)$	$\mu\lambda_1\lambda_3^{-1}(\lambda_1^2-\lambda_3^2)/[\mu\lambda_1^{-1}\lambda_3^{-1}(1-\lambda_3^2)]$
W_6	$\lambda_3-1 = -\lambda(\lambda_1-1)/(\lambda+2\mu)$	$2\mu\lambda_3^{-1}(\lambda_1-\lambda_3)/[2\mu\lambda_1^{-1}\lambda_3^{-1}(1-\lambda_3)]$
W_7	$\dfrac{2\lambda}{\lambda+\mu}j_1^\times + (j_1^\times)^2 + 4\Big(\dfrac{\mu}{\lambda+\mu} - j_1^\times\Big)A_3 = 0$	$\dfrac{2[\mu-(\lambda+\mu)j_1^\times](A_1-A_3)}{-2[\mu-(\lambda+\mu)j_1^\times]A_3}$
W_8	$B\Theta + \mu I_3^{-5/6}(2\lambda_3^2-\lambda_1^2-1)/3 = 0$	$\mu I_3^{-5/6}(\lambda_1^2-\lambda_3^2)/[\mu I_3^{-5/6}(1-\lambda_3^2)]$
W_9	$B\Theta + 2\mu I_3^{-2/3}(2\lambda_3-\lambda_1-1)/3 = 0$	$2\mu I_3^{-2/3}(\lambda_1-\lambda_3)/[2\mu I_3^{-2/3}(1-\lambda_3)]$
W_{10}	$(\lambda+2\mu)(I_3-1) + 2\mu(\lambda_1^{-2}-1) = 0$	$\mu I_3^{-1/2}(\lambda_1^2-\lambda_2^2)/[\mu I_3^{-1/2}(1-\lambda_2^2)]$

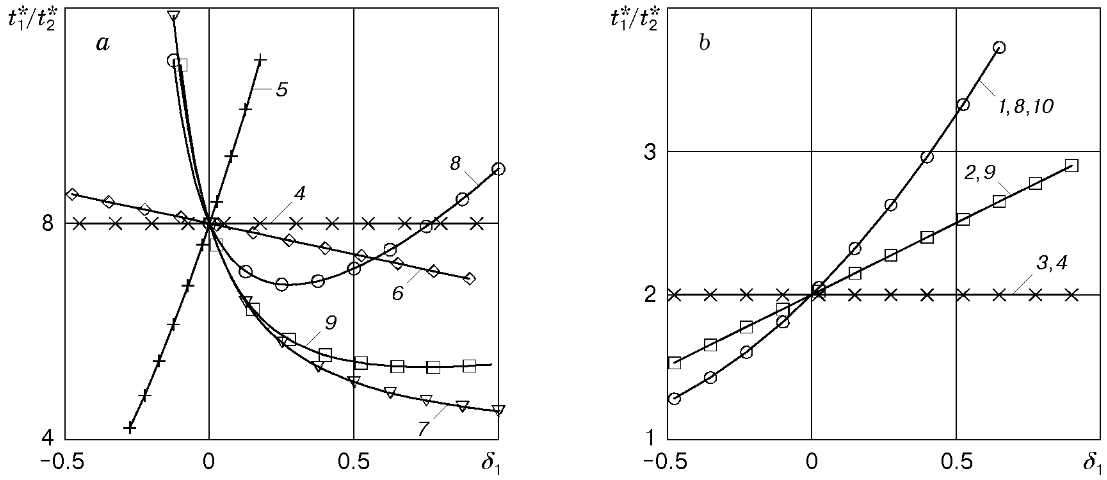

Fig. 3

An analysis of the analytical relations $\Theta(\delta_1)$ for models (5)–(7) under SBSS and NBDS has some specific features, but its results are similar to those for USS.

Conclusions. The results of this work can be summarized as follows.

1. Models (4), (8), and (9) are the best generalizations of Hooke's law within the range of μ/B considered. The advantage of model (8) over models (4) and (9) is that reduction to the principal axes of strain is not needed in boundary-value problems where an arbitrary stress–strain state is considered. As $\mu/B \to 0$, models (4), (8), and (9) become models (3), (1), and (2), respectively.

2. Models of compressible materials (5)–(7) are applicable within a limited range of the parameter μ/B. For finite strains, these models cannot be used to describe the behavior of monolithic rubber-like materials ($\mu/B \ll 1$), since they predict physically incorrect changes in volume.

3. The common property of models (5)–(7) and (10), which are applicable within a limited range of the parameter μ/B, is that the specific potential strain energy cannot be decomposed into the energy due to distortion and the energy due to the change in volume (or the latter cannot be ignored in models of incompressible materials).

4. In choosing a model, the ratio of the applied stress to the holding stress t_1^*/t_2^* measured for NBDS enables us to estimate the order of the best generalized strain measure to construct the simplest model of the behavior of incompressible and slightly compressible materials.

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